ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2025, Vol. 86, No. 3, pp. 203–216.

Stability Analysis of Non-stationary Mechanical Systems with Discontinuous Right-hand Sides

A. V. Platonov

Saint Petersburg State University, Saint Petersburg, Russia
e-mail: a.platonov@spbu.ru

Received April 14, 2024
Revised December 4, 2024
Accepted December 23, 2024

Abstract—The paper investigates the stability problem for a class of non-stationary mechanical systems under the action of linear dissipative and nonlinear potential forces. It is assumed that the system has a changeable structure. Switching between different operating modes is associated with a change of the potential of the system, as well as with discontinuities of non-stationary coefficients present in the system. Two approaches to the analysis of the stability of such systems are considered. One is related to the construction of a discontinuous Lyapunov function, the other is based on the construction of a continuous Lyapunov function. The paper also studies the effect of non-stationary perturbed forces on the stability. The peculiarity of the work is that the non-stationary parameters both in the system itself and in the perturbations can be unbounded with respect to time, or, on the contrary, they can arbitrarily approach to zero. Thus, the problem arises of comparing the rate of growth or decrease of all these non-stationarities in order to obtain conditions that guarantee the asymptotic stability of the given equilibrium position of the system.

Keywords: nonlinear non-stationary mechanical systems, switching, asymptotic stability, perturbations

DOI: 10.31857/S0005117925030021

1. INTRODUCTION

In recent decades, the theory of differential systems with discontinuous right-hand sides has been actively developing. Such systems find many practical applications in various fields of human activity. A.M. Lyapunov's methods, developed by him for analyzing the stability of continuous systems, can be successfully extended to discontinuous systems. In particular, significant results in this direction were obtained by E.A. Barbashin, A.I. Lurie, A.M. Letov, N.N. Krasovsky, V.I. Zubov, A.F. Filippov, S.V. Yemelyanov, V.M. Matrosov and many other famous scientists (see, for instance, [1]). An important class of discontinuous systems are switched systems capable to operate in different modes [2]. The activation rule of a particular mode is set by some special function called the switching law. A lot of works have been devoted to the problem of assessing the effect of switching on various dynamic characteristics of the investigated systems, including in recent years (see, for instance, [2–11] and references therein). So, currently relevant research areas related to the analysis of the influence of switching in combination with some other factors such as non-stationarity, perturbations, delay, randomness, impulse effects, complexity and hybridity of internal relationships between various variables in the system, etc. The switching laws, depending on both time and current state vector of the system, are considered. Approaches to the construction of both a single Lyapunov function for all modes and a multiple Lyapunov function formed from partial functions corresponding to different modes are actively developing. Non-stationary switched systems are of particular interest [5–7]. Here, the smooth dynamics of the system caused by the continuous change of non-stationary parameters are superimposed by sharp fluctuations caused by discontinuities of non-stationary parameters or a change in the structure of the system itself. The peculiarity here is that the number of possible operating modes of the system becomes, generally speaking, infinite. The analysis of such systems becomes more complicated if the non-stationary parameters in the system can be unbounded or inseparable from zero.

An important class of dynamic systems are mechanical systems. It is known that the presence of non-stationary parameters in mechanical systems can lead to fundamentally new dynamic effects [12–14]. In this paper, we study one class of nonlinear mechanical systems with discontinuous non-stationary parameters and switched force fields. The switching law is assumed to be time-dependent. Thus, the right-hand sides of the considered system turn out to be discontinuous with respect to time and continuous with respect to the state vector. Different ways of constructing a suitable Lyapunov function are proposed. The most natural approach here is associated with the use of separate continuous Lyapunov functions on smooth intervals of the system functioning and with the construction of an itog discontinuous Lyapunov function from them on the entire time interval. However, such an approach can lead to rather conservative stability conditions. Therefore, the paper also develops an approach to constructing a single continuous Lyapunov function for the entire hybrid system. In addition, the paper examines the effect of non-stationary perturbations on the stability of the given equilibrium position of the mechanical system. It is important to note that the desired conditions for perturbations, in general, will depend on the restrictions imposed on the switching law in the original system. The results obtained in this work can be used to analyze the robustness of non-stationary switched systems.

2. STATEMENT OF THE PROBLEM

Consider the non-stationary mechanical system under the influence of linear dissipative forces and nonlinear potential forces

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} = -\mathbf{B}(t)\dot{\mathbf{q}} - p(t)\frac{\partial \Pi_\sigma(\mathbf{q})}{\partial \mathbf{q}}. \tag{1}$$

Here $\mathbf{q} \in \mathrm{R}^n$ and $\dot{\mathbf{q}} \in \mathrm{R}^n$ are vectors of generalized coordinates and velocities, respectively; kinetic energy $T = T(\mathbf{q}, \dot{\mathbf{q}})$ of the system is set by quadratic form $T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}(\mathbf{q})\dot{\mathbf{q}}$ with a symmetric and continuously differentiable for $\mathbf{q} \in \mathrm{R}^n$ matrix $\mathbf{A}(\mathbf{q})$; $\sigma = \sigma(t) : [0, +\infty) \to S = \{1, \ldots, N\}$ is a piecewise constant function defining the switching law of potentials in the system; potentials $\Pi_s(\mathbf{q})$ are determined by continuously differentiable for $\mathbf{q} \in \mathrm{R}^n$ homogeneous functions of the order $\mu + 1$, $\mu > 1$, $s = 1, \ldots, N$; elements of symmetric matrix $\mathbf{B}(t)$ are piecewise continuous for $t \geqslant 0$; scalar function $p(t)$ is piecewise continuously differentiable for $t \geqslant 0$. Without loss of generality, we suppose that the set of break points of function $p(t)$ and its derivative is contained in the set of break points of function $\sigma(t)$ (otherwise, we combine these two sets into one, and consider this new set as a sequence of switching moments for potential forces).

Following the standard assumptions, we suppose that the kinetic energy satisfies for all $\mathbf{q} \in \mathrm{R}^n$, $\dot{\mathbf{q}} \in \mathrm{R}^n$ the estimates

$$k_1 \|\dot{\mathbf{q}}\|^2 \leqslant T(\mathbf{q}, \dot{\mathbf{q}}) \leqslant k_2 \|\dot{\mathbf{q}}\|^2,$$

$$\left\| \frac{\partial T(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \right\| \leqslant k_3 \|\dot{\mathbf{q}}\|, \quad \left\| \frac{\partial T(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} \right\| \leqslant k_4 \|\dot{\mathbf{q}}\|^2,$$

where k_1, k_2, k_3, k_4 are positive constants.

Thus, the dynamics of system (1) is influenced by both smooth changes of non-stationary parameters and sharp jumps caused by discontinuities of non-stationary parameters or a change of potential.

The purpose of the article is to establish conditions that guarantee the asymptotic stability of the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1), as well as to assess the effect on the stability of possible non-stationary perturbed forces. It should be noted that a switched system of form (1) with stationary parameters was studied in [15]. There, switching took place between a finite set of possible operating modes of the system. The presence of non-stationary parameters with, generally speaking, an infinite number of break points complicates the problem, because now the number of possible modes becomes infinite. In addition, if these non-stationary parameters can increase unboundedly or, conversely, approach to zero arbitrarily close, then the results obtained in [15] will not be applicable to such a non-stationary system.

3. USING THE DISCONTINUOUS LYAPUNOV FUNCTION

Let the sequence $\{\tau_i\}_{i=0}^{+\infty}$, where $0 = \tau_0 < \tau_1 < \ldots$, determine the switching moments of potential forces. We suppose that the number of these switching moments is finite on any finite time interval, while their total number on the interval $[0, +\infty)$ is infinite. For certainty, we assume that the elements of matrix $\mathbf{B}(t)$, as well as functions $\sigma(t)$ and $p(t)$ are right-hand continuous at their break points.

Next, we will use the following assumptions.

Assumption 1. For all $t \geqslant 0$ and $\mathbf{z} \in \mathrm{R}^n$ the inequalities

$$b_1(t)\|\mathbf{z}\|^2 \leqslant \mathbf{z}^{\mathrm{T}}\mathbf{B}(t)\mathbf{z} \leqslant b_2(t)\|\mathbf{z}\|^2$$

are valid, where $b_1(t), b_2(t)$ are piecewise continuous positive functions with positive left-hand limits at break points.

Assumption 2. Potentials $\Pi_s(\mathbf{q})$ are positive definite.

It follows from Assumption 2 that there exist positive constants c_{1s}, c_{2s} such that

$$c_{1s}\|\mathbf{q}\|^{\mu+1} \leqslant \Pi_s(\mathbf{q}) \leqslant c_{2s}\|\mathbf{q}\|^{\mu+1}, \quad s = 1, \ldots, N,$$

for all $\mathbf{q} \in \mathrm{R}^n$.

Assumption 3. Let the following conditions be fulfilled:

1) function $p(t)$ is positive for all $t \geqslant 0$, and $p(\tau_i - 0) > 0$, $i = 1, 2, \ldots$;

2) the inequality

$$-p'(t) \leqslant \frac{l}{k_2} p(t) b_1(t)$$

is valid for all $t \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots$. Here $l = \mathrm{const} \in (0, 1)$.

If parameter $p(t)$ increases at some time point t, then the inequality in condition 2) of Assumption 3 will be automatically fulfilled for this moment. Thus, this inequality impose a restriction only on the permissible rate of decrease of parameter $p(t)$.

Construct the discontinuous Lyapunov function

$$V(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{p(t)} T(\mathbf{q}, \dot{\mathbf{q}}) + \Pi_\sigma(\mathbf{q}) + r\,\gamma(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}} \frac{\partial T}{\partial \dot{\mathbf{q}}}, \tag{2}$$

where r is a positive constant, $\gamma(t)$ is a piecewise continuously differentiable positive for $t \geqslant 0$ function. As in the case of function $p(t)$, we assume that the discontinuities of function $\gamma(t)$ and its derivative can occur only at the break points of function $\sigma(t)$.

Let us differentiate function $V(t, \mathbf{q}, \dot{\mathbf{q}})$ with respect to solutions of system (1) on the intervals (τ_i, τ_{i+1}), $i = 0, 1, \ldots$. We obtain

$$\dot{V}\big|_{(1)} = -\frac{1}{p(t)}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{B}(t)\dot{\mathbf{q}} - r\gamma(t)p(t)(\mu+1)\,\|\mathbf{q}\|^{\mu-1}\,\Pi_\sigma(\mathbf{q})$$
$$- r\gamma(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}}\mathbf{B}(t)\dot{\mathbf{q}} + r\gamma(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}}\frac{\partial T}{\partial \mathbf{q}} + r\gamma(t)\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}}\frac{\partial}{\partial \mathbf{q}}\left(\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}\right)\dot{\mathbf{q}}$$
$$- \frac{p'(t)}{p^2(t)}T(\mathbf{q}, \dot{\mathbf{q}}) + r\gamma'(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}}.$$

Then for $\mathbf{q}, \dot{\mathbf{q}} \in \mathrm{R}^n$, $t \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots$, one can find the estimates

$$\frac{k_1}{p(t)}\,\|\dot{\mathbf{q}}\|^2 + c_{1\sigma}\,\|\mathbf{q}\|^{\mu+1} - r\gamma(t)k_3\,\|\dot{\mathbf{q}}\|\,\|\mathbf{q}\|^{\mu} \leqslant V(t, \mathbf{q}, \dot{\mathbf{q}})$$
$$\leqslant \frac{k_2}{p(t)}\,\|\dot{\mathbf{q}}\|^2 + c_{2\sigma}\,\|\mathbf{q}\|^{\mu+1} + r\gamma(t)k_3\,\|\dot{\mathbf{q}}\|\,\|\mathbf{q}\|^{\mu},$$
$$\dot{V}\big|_{(1)} \leqslant -\frac{b_1(t)}{p(t)}\,\|\dot{\mathbf{q}}\|^2 - r\gamma(t)p(t)(\mu+1)c_{1\sigma}\,\|\mathbf{q}\|^{2\mu}$$
$$+ r\gamma(t)b_2(t)\,\|\dot{\mathbf{q}}\|\,\|\mathbf{q}\|^{\mu} + r\gamma(t)k_4\,\|\dot{\mathbf{q}}\|^2\,\|\mathbf{q}\|^{\mu} + r\gamma(t)k_3 a\,\|\dot{\mathbf{q}}\|^2\,\|\mathbf{q}\|^{\mu-1}$$
$$+ \max\left\{0;\ \frac{-p'(t)k_2}{p^2(t)}\right\}\|\dot{\mathbf{q}}\|^2 + r|\gamma'(t)|k_3\,\|\dot{\mathbf{q}}\|\,\|\mathbf{q}\|^{\mu}.$$

Here a is a positive constant chosen so that for all $\mathbf{q} \in \mathrm{R}^n$ the condition $\left\|\frac{\partial}{\partial \mathbf{q}}\left(\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}\right)\right\| \leqslant a\|\mathbf{q}\|^{\mu-1}$ takes place.

For further analysis, we will need one auxiliary result.

Let the function

$$W(t, \mathbf{z}) = -\alpha(t)z_1^p - \beta(t)z_2^q + \delta(t)z_1^u z_2^v$$

be given, where $t \geqslant 0$, $\mathbf{z} = (z_1, z_2)^{\mathrm{T}}$, $z_1, z_2 \in [0, +\infty)$; p and q are positive constants; u and v are nonnegative constants, $u + v > 0$; functions $\alpha(t)$, $\beta(t)$, $\delta(t)$ are piecewise continuous and positive for $t \geqslant 0$.

Lemma 1 [16]. *Let the inequality*

$$\frac{u}{p} + \frac{v}{q} > 1 \tag{3}$$

be valid and there be a constant ε satisfying the condition

$$\max\{0;\ v - q\} \leqslant \varepsilon \leqslant \min\left\{v;\ v - \frac{(p-u)q}{p}\right\},$$

such that function

$$\frac{\delta(t)}{\alpha(t)}\left(\frac{\alpha(t)}{\beta(t)}\right)^{\frac{v-\varepsilon}{q}} \tag{4}$$

is bounded for $t \geqslant 0$. Then for any $M \in (0, 1)$ one can choose $H > 0$ such that the estimate

$$W(t, \mathbf{z}) \leqslant M\left(-\alpha(t)z_1^p - \beta(t)z_2^q\right) \tag{5}$$

will be valid for $t \geqslant 0$, $\|\mathbf{z}\| < H$.

Remark 1. It is easy to see that if function $\alpha(t)/\beta(t)$ is bounded from above for $t \geqslant 0$, then it is sufficient to check the boundedness of function (4) in conditions of Lemma 1 only for $\varepsilon = \max\{0; v - q\}$. If function $\alpha(t)/\beta(t)$ is bounded from below by a positive constant for $t \geqslant 0$, then it is sufficient to check the boundedness of function (4) only for $\varepsilon = \min\{v; v - (p-u)q/p\}$. If inequality (3) is replaced by equality and we assume that function (4) is bounded from above on the interval $[0, +\infty)$ by a sufficiently small positive constant, then for some $M \in (0,1)$, estimate (5) can be obtained for all $t \geqslant 0$, $z_1, z_2 \in [0, +\infty)$.

We apply Lemma 1 to the estimates obtained earlier for Lyapunov function (2) and its derivative with respect to solutions of system (1). For any values of $a_1 \in (0,1)$ and $a_2 > 1$ one can choose function $\gamma(t)$ and positive constants r, a_3, a_4, H so that

$$a_1 \left(\frac{k_1}{p(t)} \|\dot{\mathbf{q}}\|^2 + c_{1\sigma} \|\mathbf{q}\|^{\mu+1} \right) \leqslant V(t, \mathbf{q}, \dot{\mathbf{q}}) \leqslant a_2 \left(\frac{k_2}{p(t)} \|\dot{\mathbf{q}}\|^2 + c_{2\sigma} \|\mathbf{q}\|^{\mu+1} \right) \tag{6}$$

for $t \geqslant 0$, $\|(\mathbf{q}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}\| < H$,

$$\dot{V}\big|_{(1)} \leqslant -a_3 \left(\frac{b_1(t)}{p(t)} \|\dot{\mathbf{q}}\|^2 + r\gamma(t)p(t)(\mu+1)c_{1\sigma} \|\mathbf{q}\|^{2\mu} \right) \leqslant -a_4 \lambda(t) V^{1+\xi}(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{7}$$

for $t \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots$, $\|(\mathbf{q}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}\| < H$. Here $\mathbf{z} = \dot{\mathbf{q}}/\sqrt{p(t)}$, $\xi = (\mu-1)/(\mu+1)$, $\lambda(t) = \min\{b_1(t); \gamma(t)p(t)\}$. Really, according to Lemma 1, in order to obtain estimates (6), (7) it is required that the functions

$$\gamma(t)\sqrt{p(t)}, \qquad \gamma(t)\frac{b_2^2(t)}{b_1(t)}, \qquad \frac{\gamma(t)\,p(t)}{b_1(t)}, \tag{8}$$

$$\frac{(\gamma'(t))^2}{\gamma(t) b_1(t)} \tag{9}$$

be bounded for $t \geqslant 0$. To achieve the boundedness of functions (8), it is sufficient to construct function $\gamma(t)$, according to the condition

$$0 \leqslant \gamma(t) \leqslant N \, \min\left\{ 1/\sqrt{p(t)};\, b_1(t)/b_2^2(t);\, b_1(t)/p(t) \right\} \quad \text{for} \quad t \geqslant 0, \tag{10}$$

where N is a positive constant. The boundedness of functions (9) will be guaranteed, for instance, if to construct function $\gamma(t)$ in the form of piecewise constant function (then we have $\gamma'(t) = 0$ for $t \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots$).

Assume

$$\varkappa_i = \frac{a_2}{a_1} \max\left\{ \frac{k_2\, p(\tau_i - 0)}{k_1\, p(\tau_i)};\, \frac{c_{2\sigma(\tau_i)}}{c_{1\sigma(\tau_i - 0)}} \right\}, \quad i = 1, 2, \ldots.$$

Then for $\|(\mathbf{q}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}\| < H$ we have

$$V(\tau_i, \mathbf{q}, \dot{\mathbf{q}}) \leqslant \varkappa_i V(\tau_i - 0, \mathbf{q}, \dot{\mathbf{q}}), \quad i = 1, 2, \ldots. \tag{11}$$

For any $t \geqslant t_0 \geqslant 0$ one can find positive integer m and nonnegative integer k such that $t_0 \in [\tau_{m-1}, \tau_m)$, $t \in [\tau_{m-1+k}, \tau_{m+k})$. So, we have that the value of $m = m(t_0)$ is defined by the choice of t_0, whereas the value of $k = k(t_0, t)$ is equal to the number of switching moments of potential in the system on the interval $[t_0, t]$.

Construct auxiliary function $\psi(t_0, t)$ by the following formulas:

$$\psi(t_0, t) = \int\limits_{t_0}^{t} \lambda(\tau)\, d\tau \quad \text{for} \quad k = 0,$$

$$\psi(t_0, t) = (\varkappa_m \ldots \varkappa_{m-1+k})^{-\xi} \int\limits_{t_0}^{\tau_m} \lambda(\tau)\, d\tau$$

$$+ \sum_{j=1}^{k-1} (\varkappa_{m+j} \ldots \varkappa_{m-1+k})^{-\xi} \int\limits_{\tau_{m-1+j}}^{\tau_{m+j}} \lambda(\tau)\, d\tau + \int\limits_{\tau_{m-1+k}}^{t} \lambda(\tau)\, d\tau \quad \text{for} \quad k = 1, 2, \ldots .$$

Function $\psi(t_0, t)$ is positive and piecewise continuous on the interval $[t_0, +\infty)$. Switching moments of potential forces are, in general, the break points of this function. Note that function $\psi(t_0, t)$ for some fixed value of t_0 differs from function $\psi(0, t)$ by shifting by a constant dependent on the choice of t_0. For the constructing function $\psi(0, t)$ one should set in the formulas written out above $t_0 = 0$, $m = m(0) = 1$, and define $k = k(0, t)$ as the number of switching moments on the interval $[0, t]$.

Consider the solution $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ of system (1), starting at time moment $t_0 \geqslant 0$ from the point $(\mathbf{q}_0, \dot{\mathbf{q}}_0)$ such that $\|(\mathbf{q}_0^{\mathrm{T}}, \mathbf{z}_0^{\mathrm{T}})^{\mathrm{T}}\| < H$ (here $\mathbf{z}_0 = \dot{\mathbf{q}}_0/\sqrt{p(t_0)}$). By integrating differential inequalities (7) and using conditions (11), one gets that, if on the interval $[t_0, t]$ the solution remains in the domain $\|(\mathbf{q}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}\| < H$, then the estimates

$$V^{-\xi}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \geqslant V^{-\xi}(t_0, \mathbf{q}_0, \dot{\mathbf{q}}_0) + a_4\, \xi\, \psi(t_0, t) \quad \text{for} \quad k = 0,$$

$$V^{-\xi}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \geqslant (\varkappa_m \ldots \varkappa_{m-1+k})^{-\xi}\, V^{-\xi}(t_0, \mathbf{q}_0, \dot{\mathbf{q}}_0) + a_4\, \xi\, \psi(t_0, t) \quad \text{for} \quad k = 1, 2, \ldots,$$

hold.

Then, taking into account inequalities (6) and doing the same reasoning as in proving Theorem 1 from [17], we come to the following result.

Theorem 1. *Let Assumptions* 1–3 *be fulfilled and estimates* (6), (7), (11) *be constructed. If* $\psi(0, t) \to +\infty$ *and* $p^{-\xi}(t)\psi(0, t) \to +\infty$ *as* $t \to +\infty$, *then the equilibrium position* $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ *of system* (1) *is asymptotically stable.*

Remark 2. Conditions of the asymptotic stability formulated in Theorem 1 can be replaced by a coarser but simpler discrete version. Denote

$$\lambda_i = \inf_{[\tau_i, \tau_{i+1}]} \lambda(t), \quad p_k = \sup_{[\tau_k, \tau_{k+1}]} p(t), \quad \psi_k = \sum_{i=0}^{k-1} (\varkappa_{i+1} \ldots \varkappa_k)^{-\xi}\, \lambda_i T_i, \quad k = 1, 2, \ldots .$$

Here $T_i = \tau_{i+1} - \tau_i$, $i = 0, 1, \ldots$. Then, for the asymptotic stability of the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1) it is sufficient to fulfill the conditions: $\psi_k \to +\infty$ and $p_k^{-\xi}\psi_k \to +\infty$ as $k \to +\infty$. Thus, found conditions of the asymptotic stability are determined through relations linking the lengths of the intervals between successive switches of potential forces in system (1), the magnitude of jumps in the Lyapunov function caused by discontinuities of parameter $p(t)$ and potential changes, as well as the rate of change of functions $b_1(t)$, $b_2(t)$, $p(t)$.

Now, let us suppose that considered system is under influence of some perturbed forces:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} = -\mathbf{B}(t)\dot{\mathbf{q}} - p(t)\frac{\partial \Pi_\sigma(\mathbf{q})}{\partial \mathbf{q}} + \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}). \tag{12}$$

Assume that function $\mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ is defined in the region

$$t \geqslant 0, \quad \|(\mathbf{q}^{\mathrm{T}}, \dot{\mathbf{q}}^{\mathrm{T}})^{\mathrm{T}}\| < \Delta \tag{13}$$

and satisfies there the conditions guaranteeing the existence, uniqueness of the solutions of system (12) and their continuous dependence on the initial data.

Let estimate

$$\|\mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}})\| \leqslant h(t)\|\mathbf{q}\|^{\omega} + g(t)\|\dot{\mathbf{q}}\|^{\eta}$$

be valid in region (13), where ω and η are positive constants, $h(t)$ and $g(t)$ are nonnegative piecewise continuous functions. Note that functions $h(t)$ and $g(t)$ can be unbounded on the interval $[0,+\infty)$.

Sufficient conditions of the asymptotic stability of the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ of system (12) can be obtained, as before, by applying Lemma 1 to Lyapunov function (2) and its derivative with respect to solutions of system (12).

Corollary 1. *Let conditions of Theorem* 1 *be fulfilled. If*

$$\omega > \mu, \qquad \eta > 1, \tag{14}$$

and, moreover, functions

$$\chi_1(t) = \frac{h(t)}{\gamma^{1/2}(t)\, b_1^{1/2}(t)\, p(t)}, \qquad \chi_2(t) = \frac{g(t)\, p^{(\eta-1)/2}(t)}{b_1(t)} \tag{15}$$

are bounded on the interval $[0,+\infty)$, *then the equilibrium position* $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ *of system* (12) *is asymptotically stable.*

Really, with the assumptions made in Corollary 1, the application of Lemma 1 will again lead us to estimates of form (6), (7), (11), and accordingly, the fulfillment of the conditions of Theorem 1 will guarantee the required property for the perturbed system.

If some of functions (15) are unbounded on the interval $[0,+\infty)$, then the approach proposed in [18] can be used to establish the conditions of the asymptotic stability. In [18], continuous Lyapunov functions were used. We show that the application of discontinuous Lyapunov functions is also acceptable within the framework of this approach.

Corollary 2. *Let conditions of Theorem* 1 *be fulfilled. If inequalities* (14) *are valid, and the tendencies*

$$\widetilde{\chi}_1(t) = \chi_1(t)\,\psi^{-(\omega-\mu)/(\mu-1)}(0,t) \to 0 \quad \text{as} \quad t\to+\infty,$$
$$\widetilde{\chi}_2(t) = \chi_2(t)\,\psi^{-(\eta-1)/(2\xi)}(0,t) \to 0 \quad \text{as} \quad t\to+\infty$$

hold, then the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ *of system* (12) *is asymptotically stable.*

The proof of Corollary 2 is given in the Appendix.

Example 1. Let Assumptions 1–3 be fulfilled, and there exist positive constants L_1, L_2, L_3, L_4 and $\hat{t}$ such that the inequalities

$$L_1\, t \leqslant p(t) \leqslant L_2\, t, \quad L_3\sqrt{t} \leqslant b_1(t) \leqslant b_2(t) \leqslant L_4\sqrt{t} \quad \text{for} \quad t \geqslant \hat{t}$$

are valid. Then, by choosing $\gamma(t) = N/\sqrt{\tau_{i+1}}$ for $t\in[\tau_i,\tau_{i+1})$ ($N=\text{const}>0$), $i=0,1,\ldots$, one gets that functions (8), (9) will be bounded. Note that, in the considered case, $p(\tau_i-0)/p(\tau_i) \leqslant L_2/L_1$ for all values of index i such that $\tau_i \geqslant \hat{t}$. Hence, there exists $\varkappa>1$ such that $\varkappa_i \leqslant \varkappa$, $i=1,2,\ldots$. We have

$$\lambda(t) \geqslant \min\left\{L_3\sqrt{t};\ NL_1\, t/\sqrt{\tau_{i+1}}\right\} \geqslant \min\{L_3\,;NL_1\}\, t/\sqrt{\tau_{i+1}}$$

for $t\in[\tau_i,\tau_{i+1})$, $\tau_i\geqslant\hat{t}$. Applying Theorem 1, we obtain that for the asymptotic stability of the equilibrium position $\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}$ of system (1) it is sufficient to fulfill the condition

$$\frac{1}{\tau_{k+1}^{\xi}}\sum_{i=0}^{k-1}\varkappa^{-\xi(k-i)}\frac{\tau_{i+1}+\tau_i}{\sqrt{\tau_{i+1}}}\,T_i \to +\infty \quad \text{as} \quad k\to+\infty.$$

Assume, for instance, that $T_i = \text{const} > 0$, $i = 0, 1, \ldots$ (i.e. switching of the potential forces occurs at fixed time intervals). Then, the found condition of the asymptotic stability will be fulfilled, if $\xi < 1/2$, i.e. if $\mu < 3$. In this case, there is a constant $L_5 > 0$ such that $\psi(0,t) \geqslant L_5\sqrt{t}$ for $t \geqslant \hat{t}$. Therefore, according to Corollary 2, the asymptotic stability of the given equilibrium position will be preserved for perturbed system (12), if inequalities (14) are valid and tendencies

$$h(t)t^{-\frac{\omega+\mu-2}{2(\mu-1)}} \to 0, \quad g(t)t^{-\frac{\eta+4\xi-2\xi\eta-1}{4\xi}} \to 0 \quad \text{as} \quad t \to +\infty$$

hold.

4. USING THE CONTINUOUS LYAPUNOV FUNCTION

The presence of multipliers $\varkappa_i$, $i = 1, 2, \ldots$, in estimates (11) is caused by discontinuities of Lyapunov function (2), and, in Theorem 1, it can lead to rather conservative restrictions on the switching law of potential forces. In this section, we construct a continuous Lyapunov function for the study of the asymptotic stability of the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of systems (1) and (12).

Let us replace Assumption 2 with a weaker one.

Assumption 4. Potential $\Pi_1(\mathbf{q})$ is positive definite.

Taking into account Assumption 4, we obtain that there exist positive constants c_{11}, c_{2s}, c_{3s}, $s = 1, \ldots, N$, such that for all $\mathbf{q} \in \mathrm{R}^n$ the inequalities

$$c_{11}\|\mathbf{q}\|^{\mu+1} \leqslant \Pi_1(\mathbf{q}) \leqslant c_{21}\|\mathbf{q}\|^{\mu+1}, \quad |\Pi_s(\mathbf{q})| \leqslant c_{2s}\|\mathbf{q}\|^{\mu+1}, \quad s = 2, \ldots, N,$$

$$\left\|\frac{\partial \Pi_s(\mathbf{q})}{\partial \mathbf{q}}\right\| \leqslant c_{3s}\|\mathbf{q}\|^{\mu}, \quad s = 1, \ldots, N,$$

are valid.

For example, one can suppose that some stabilizing control is active for $\sigma(t) = 1$, and this control is deactivated for $\sigma(t) \neq 1$. Then it is necessary to find a ratio between the lengths of the intervals of activity of the control and the lengths of the intervals for deactivating it, which guarantees the preservation of the asymptotic stability of the given equilibrium position. Let the sequence $\{\tilde{\tau}_i\}_{i=0}^{+\infty}$, where $0 = \tilde{\tau}_0 < \tilde{\tau}_1 < \ldots$, set the switching moments in which the value of $\sigma(t)$ changes from 1 to some other, or vice versa. For the definiteness, we assume that $\sigma(t) = 1$ for $t \in [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, and $\sigma(t) \neq 1$ for $t \in [\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$; $j = 0, 1, \ldots$.

Instead of Assumption 3, we will use the following assumption.

Assumption 5. Let the following conditions be fulfilled:

1) function $p(t)$ is positive for $t \in [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, and $p(\tilde{\tau}_{2j+1} - 0) > 0$, $j = 0, 1, \ldots$;

2) for all $t \in (\tilde{\tau}_{2j}, \tilde{\tau}_{2(j+1)})$ the inequality

$$-p'(t) \leqslant \frac{l}{k_2} p(t) b_1(t)$$

is valid and, moreover,

$$p(\tilde{\tau}_{2j+1} - 0) - p(\tilde{\tau}_{2(j+1)}) \leqslant \frac{l}{k_2}\left(\tilde{\tau}_{2(j+1)} - \tilde{\tau}_{2j+1}\right) p(\tilde{\tau}_{2(j+1)}) \inf_{(\tilde{\tau}_{2j+1};\, \tilde{\tau}_{2(j+1)})} b_1(t).$$

Here $l = \text{const} \in (0, 1)$, $j = 0, 1, \ldots$.

Condition 2) in Assumption 5, imposes a restriction on the permissible rate of decrease of parameter $p(t)$ at $\sigma(t) = 1$, as well as on the permissible decrease of the value of $p(t)$ from the moment of deactiving the mode $\sigma(t) = 1$ until the moment of activating this mode again.

Construct continuous function $\widetilde{p}(t)$ such that $\widetilde{p}(t) = p(t)$ for $t \in [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, $j = 0, 1, \ldots$. On the intervals $[\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$, one can determine function $\widetilde{p}(t)$ as linear (i.e. the continuous pieces of function $p(t)$ corresponding to the mode $\sigma(t) = 1$ are "glued" using straight lines).

Let us construct continuous Lyapunov function

$$\widetilde{V}(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{\widetilde{p}(t)} T(\mathbf{q}, \dot{\mathbf{q}}) + \Pi_1(\mathbf{q}) + r\,\tilde{\gamma}(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}} \frac{\partial T}{\partial \dot{\mathbf{q}}}, \tag{16}$$

where r is a positive constant, $\tilde{\gamma}(t)$ is a continuous piecewise differentiable positive for $t \geqslant 0$ function.

On the intervals $[\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, $j = 0, 1, \ldots$, we find function $\gamma(t)$, as in the previous section of the paper, for instance, in form of constants satisfying estimate (10). Define function $\tilde{\gamma}(t)$ by continuously "gluing" the constructed pieces of found function $\gamma(t)$ so that the condition

$$0 \leqslant \tilde{\gamma}(t) \leqslant N \min\left\{1/\sqrt{\widetilde{p}(t)};\; b_1(t)/\widetilde{p}(t)\right\} \quad \text{for} \quad t \geqslant 0$$

is fulfilled. On the intervals $[\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, $j = 0, 1, \ldots$, this condition follows from (10), therefore, it is enough to "glue" $\tilde{\gamma}(t)$ in such a way that the specified condition is preserved on the intervals $[\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$. Given the continuity of function $\widetilde{p}(t)$ and the piecewise continuity of function $b_1(t)$, this can obviously be done if additional requirements:

$$\gamma(\tilde{\tau}_{2j+1} - 0) \leqslant N\,\frac{b_1(\tilde{\tau}_{2j+1})}{p(\tilde{\tau}_{2j+1} - 0)}, \quad \gamma(\tilde{\tau}_{2(j+1)}) \leqslant N\,\frac{b_1(\tilde{\tau}_{2(j+1)} - 0)}{p(\tilde{\tau}_{2(j+1)})}, \quad j = 0, 1, \ldots,$$

are imposed on the choice of $\gamma(t)$.

Then, applying Lemma 1, as before, we get the estimates

$$\tilde{a}_1\left(\frac{k_1}{\widetilde{p}(t)}\,\|\dot{\mathbf{q}}\|^2 + c_{11}\,\|\mathbf{q}\|^{\mu+1}\right) \leqslant \widetilde{V}(t, \mathbf{q}, \dot{\mathbf{q}}) \leqslant \tilde{a}_2\left(\frac{k_2}{\widetilde{p}(t)}\,\|\dot{\mathbf{q}}\|^2 + c_{21}\,\|\mathbf{q}\|^{\mu+1}\right), \tag{17}$$

for $t \geqslant 0$, $\|(\mathbf{q}^{\mathrm{T}}, \widetilde{\mathbf{z}}^{\mathrm{T}})^{\mathrm{T}}\| < \widetilde{H}_1$, and

$$\dot{\widetilde{V}}|_{(1),(\sigma=1)} \leqslant -\tilde{a}_3\left(\frac{b_1(t)}{p(t)}\,\|\dot{\mathbf{q}}\|^2 + r\gamma(t)p(t)(\mu+1)c_{11}\,\|\mathbf{q}\|^{2\mu}\right) \leqslant -\tilde{a}_4\lambda(t)\widetilde{V}^{1+\xi}(t, \mathbf{q}, \dot{\mathbf{q}}), \tag{18}$$

for $t \in (\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, $j = 0, 1, \ldots$, $\|(\mathbf{q}^{\mathrm{T}}, \widetilde{\mathbf{z}}^{\mathrm{T}})^{\mathrm{T}}\| < \widetilde{H}_1$. Here $\widetilde{\mathbf{z}} = \dot{\mathbf{q}}/\sqrt{\widetilde{p}(t)}$, $\widetilde{H}_1$, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$, $\tilde{a}_4$ are positive constants, $\lambda(t) = \min\{b_1(t);\, \gamma(t)p(t)\}$, $\xi = (\mu - 1)/(\mu + 1)$.

Differentiate Lyapunov function (16) with respect to solutions of system (1) on the intervals $(\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$. We obtain

$$\begin{aligned}\dot{\widetilde{V}}|_{(1),(\sigma\neq1)} &= -\frac{1}{\widetilde{p}(t)}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{B}(t)\dot{\mathbf{q}} - r\tilde{\gamma}(t)p(t)(\mu+1)\,\|\mathbf{q}\|^{\mu-1}\,\Pi_\sigma(\mathbf{q})\\
&- r\tilde{\gamma}(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}}\mathbf{B}(t)\dot{\mathbf{q}} + r\tilde{\gamma}(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}}\frac{\partial T}{\partial \mathbf{q}} + r\tilde{\gamma}(t)\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^{\mathrm{T}}\frac{\partial}{\partial \mathbf{q}}\left(\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}\right)\dot{\mathbf{q}}\\
&- \frac{\widetilde{p}'(t)}{\widetilde{p}^2(t)}T(\mathbf{q}, \dot{\mathbf{q}}) + r\tilde{\gamma}'(t)\,\|\mathbf{q}\|^{\mu-1}\,\mathbf{q}^{\mathrm{T}}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{p(t)}{\widetilde{p}(t)}\dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial \Pi_\sigma(\mathbf{q})}{\partial \mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}}\frac{\partial \Pi_1(\mathbf{q})}{\partial \mathbf{q}}.\end{aligned}$$

Then for $\mathbf{q}, \dot{\mathbf{q}} \in \mathrm{R}^n$, $t \in (\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$, taking into account Assumption 5, we have the estimate

$$\begin{aligned}\dot{\widetilde{V}}|_{(1),(\sigma\neq1)} &\leqslant -(1-l)\frac{b_1(t)}{\widetilde{p}(t)}\|\dot{\mathbf{q}}\|^2 + r\tilde{\gamma}(t)|p(t)|(\mu+1)c_{2\sigma}\,\|\mathbf{q}\|^{2\mu}\\
&+ r\tilde{\gamma}(t)k_4\,\|\dot{\mathbf{q}}\|^2\,\|\mathbf{q}\|^{\mu} + r\tilde{\gamma}(t)k_3a\,\|\dot{\mathbf{q}}\|^2\,\|\mathbf{q}\|^{\mu-1}\\
&+ \left(r\tilde{\gamma}(t)b_2(t) + r|\tilde{\gamma}'(t)|k_3 + \frac{|p(t)|}{\widetilde{p}(t)}c_{3\sigma} + c_{31}\right)\|\dot{\mathbf{q}}\|\|\mathbf{q}\|^{\mu}.\end{aligned}$$

Next, use the Jensen inequality

$$z_1^u z_2^v = \left(\frac{z_1}{\theta}\right)^u \left(\theta^{u/v} z_2\right)^v \leqslant u\frac{z_1}{\theta} + v\theta^{u/v} z_2,$$

which is valid for any positive values of z_1, z_2, u, v, θ, if $u + v = 1$.

Choose some constant $L > 0$ satisfying the condition

$$-(1 - l) + 1/(2L) < 0.$$

Then there exist $\widetilde{H}_2 > 0$ and $\tilde{a}_5 > 0$ such that for $t \in (\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$, $\|(\mathbf{q}^{\mathrm{T}}, \widetilde{\mathbf{z}}^{\mathrm{T}})^{\mathrm{T}}\| < \widetilde{H}_2$ the following estimates are valid:

$$\dot{\widetilde{V}}\big|_{(1),(\sigma\neq 1)} \leqslant \widetilde{\lambda}(t)\, \|\mathbf{q}\|^{2\mu} \leqslant \tilde{a}_5 \widetilde{\lambda}(t) \widetilde{V}^{1+\xi}(t, \mathbf{q}, \dot{\mathbf{q}}). \tag{19}$$

Here

$$\widetilde{\lambda}(t) = r\tilde{\gamma}(t)|p(t)|(\mu + 1)c_{2\sigma} + \frac{L}{2}\left(r\tilde{\gamma}(t)b_2(t) + r|\tilde{\gamma}'(t)|k_3 + \frac{|p(t)|}{\widetilde{p}(t)}c_{3\sigma} + c_{31}\right)^2 \frac{\widetilde{p}(t)}{b_1(t)}.$$

Construct function $\Lambda(t)$ according to the rule: $\Lambda(t) = \tilde{a}_4\lambda(t)$ for $t \in [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$, $\Lambda(t) = -\tilde{a}_5\widetilde{\lambda}(t)$ for $t \in [\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$. Denote $\Psi(t_0, t) = \int_{t_0}^{t} \Lambda(\tau)\, d\tau$ for $t \geqslant t_0 \geqslant 0$. Then we have the theorem.

Theorem 2. *Let Assumptions* 1, 4, 5 *be fulfilled and estimates* (17)–(19) *be constructed. If* $\Psi(0, t) \to +\infty$ *and* $\widetilde{p}^{-\xi}(t)\Psi(0, t) \to +\infty$ *as* $t \to +\infty$*, then the equilibrium position* $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ *of system* (1) *is asymptotically stable.*

Remark 3. All constants present in formulas (17)–(19) can be evaluated and chosen in a simple way. Thus, the verification of the asymptotic stability conditions determined by Theorem 2 reduces to the analysis of the behavior of the given functions $p(t)$, $b_1(t)$, $b_2(t)$, as well as constructed auxiliary functions $\widetilde{p}(t)$, $\gamma(t)$, $\tilde{\gamma}(t)$. As in the previous section of the article (see Remark 2), to facilitate calculations, all these functions can be coarsened with constants on each from the intervals $[\tilde{\tau}_i, \tilde{\tau}_{i+1})$, $i = 0, 1, \ldots$. Then we come to a coarser, but simpler discrete version of the asymptotic stability conditions.

Example 2. Let Assumptions 1, 4, 5 be fulfilled, and there exist positive constants L_1, L_2, L_3, L_4 and $\hat{t}$ such that the inequalities

$$L_1\, t \leqslant p(t) \leqslant L_2\, t \quad \text{for} \quad t \in [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1}), \quad \tilde{\tau}_{2j} \geqslant \hat{t},$$
$$L_3 \sqrt{t} \leqslant b_1(t) \leqslant b_2(t) \leqslant L_4 \sqrt{t} \quad \text{for} \quad t \geqslant \hat{t}$$

are valid. Then we obtain that $L_1 t \leqslant \widetilde{p}(t) \leqslant L_2\, t$ for $t \geqslant \hat{t}$. Choose $\gamma(t) = N/\sqrt{\tilde{\tau}_{2j+1}}$ for $t \in [\tilde{\tau}_{2j}, \tilde{\tau}_{2j+1})$ ($N = \mathrm{const} > 0$), $j = 0, 1, \ldots$. In the considered case, for the determination of continuous function $\tilde{\gamma}(t)$ it is sufficient "to glue" constructed pieces of function $\gamma(t)$ by straight lines on the intervals $[\tilde{\tau}_{2j+1}, \tilde{\tau}_{2(j+1)})$, $j = 0, 1, \ldots$. Then, we get estimates (17)–(19). Constants in these estimates can be found for concrete given system (1) using both analytical and numerical methods.

Now, consider perturbed system (12).

Corollary 3. *Let conditions of Theorem 2 be fulfilled. If inequalities* (14) *are valid, and the tendencies*

$$\delta_1(t) = \frac{h(t)}{\widetilde{p}^{1/2}(t)\, b_1(t)} \Psi^{-(\omega-\mu)/(\mu-1)}(0,t) \to 0 \quad as \quad t \to +\infty,$$

$$\widetilde{\delta}_1(t) = \Psi^{-1}(0,t) \int\limits_{\hat{T}}^{t} \delta_1(\tau)\, b_1(\tau)\, d\tau \to 0 \quad as \quad t \to +\infty,$$

$$\delta_2(t) = \frac{g(t)\, \widetilde{p}^{(\eta-1)/2}(t)}{b_1(t)} \Psi^{-(\eta-1)/2\xi}(0,t) \to 0 \quad as \quad t \to +\infty,$$

$$\widetilde{\delta}_2(t) = \Psi^{-1}(0,t) \int\limits_{\hat{T}}^{t} \delta_2(\tau)\, b_1(\tau)\, d\tau \to 0 \quad as \quad t \to +\infty$$

hold (*here* $\hat{T}$ *is a positive constant such that* $\Psi(0,t) > 0$ *for* $t \geqslant \hat{T}$), *then the equilibrium position* $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ *of system* (12) *is asymptotically stable.*

The proof of Corollary 3 is given in the Appendix.

5. CONCLUSION

The paper considers various approaches to assessing the effect of switching, caused by both structural changes in the mechanical system and discontinuities of non-stationary coefficients, on the stability of a given equilibrium position. Since the choice of a suitable Lyapunov function, as a rule, is based on the structure and coefficients of the system, this leads us to construct, generally speaking, a discontinuous Lyapunov function. Taking into account the jumps made by the Lyapunov function at the break points can impose too strict restrictions on the permissible switching laws. Therefore, the possibility of constructing a continuous Lyapunov function is of particular interest. In addition, in the work, the effect on the stability of possible non-stationary perturbations acting on the system was investigated. It is shown that the known methods previously applied to smooth systems can be adapted to discontinuous systems. Note that the restrictions on the perturbations depend, generally speaking, on the restrictions on the switching law in the initial mechanical system.

FUNDING

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

CONFLICT OF INTEREST

The author declares that he has no conflict of interest.

APPENDIX

A.1. PROOF OF COROLLARY 2

Find the derivative of Lyapunov function (2) with respect to solutions of system (12) for $t \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots$. If $\|(\mathbf{q}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}\| < H$ and $\|(\mathbf{q}^{\mathrm{T}}, \dot{\mathbf{q}}^{\mathrm{T}})^{\mathrm{T}}\| < \Delta$, then we obtain

$$\dot{V}\big|_{(12)} \leqslant -a_3 \left(\frac{b_1(t)}{p(t)} \|\dot{\mathbf{q}}\|^2 + r\gamma(t)p(t)(\mu+1)c_{1\sigma} \|\mathbf{q}\|^{2\mu} \right)$$
$$+ \left(\frac{k_3}{p(t)} \|\dot{\mathbf{q}}\| + r\gamma(t)\|\mathbf{q}\|^{\mu} \right) \left(h(t)\|\mathbf{q}\|^{\omega} + g(t)\|\dot{\mathbf{q}}\|^{\eta} \right).$$

Let us show that constants $D_0 > 0$, $D > 0$ and $\hat{t} \geqslant 0$ can be chosen so that if

$$t_0 \geqslant \hat{t}, \qquad \frac{1}{p(t_0)}\,\|\dot{\mathbf{q}}(t_0)\|^2 + \|\mathbf{q}(t_0)\|^{\mu+1} < D_0\,\psi^{-1/\xi}(0,t_0), \tag{A.1}$$

then

$$\frac{1}{p(t)}\,\|\dot{\mathbf{q}}(t)\|^2 + \|\mathbf{q}(t)\|^{\mu+1} < D\,\psi^{-1/\xi}(0,t) \quad \text{for} \quad t \geqslant t_0. \tag{A.2}$$

Here $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ is a solution of system (12). Really, set the specified constants, according to the conditions

$$D_0 \leqslant D, \quad -(a_2\,\bar{c}\,D_0)^{-\xi} + (a_4\xi)/2 < 0, \quad -(a_1\,\hat{c}\,D)^{-\xi} + (a_4\xi)/2 > 0, \tag{A.3}$$

$$D^{2/(\mu+1)}\psi^{-2/(\mu-1)}(0,t) + D\,p(t)\,\psi^{-1/\xi}(0,t) < \Delta^2 \quad \text{for} \quad t \geqslant \hat{t}, \tag{A.4}$$

$$D^{2/(\mu+1)}\psi^{-2/(\mu-1)}(0,t) + D\,\psi^{-1/\xi}(0,t) < H^2 \quad \text{for} \quad t \geqslant \hat{t}, \tag{A.5}$$

$$\max\{\widetilde{\chi}_1(t);\ \widetilde{\chi}_2(t)\} < d \quad \text{for} \quad t \geqslant \hat{t}, \tag{A.6}$$

where $\bar{c} = \max\{k_2;\, c_{21};\, \ldots;\, c_{2N}\}$, $\hat{c} = \min\{k_1;\, c_{11};\, \ldots;\, c_{1N}\}$, d is a positive constant. Conditions (A.3)–(A.6) are compatible. At first, one can choose constants D_0 and D according to inequalities (A.3), then constant $\hat{t}$ can be chosen according to conditions (A.4)–(A.6).

Let the initial data of the solution $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ satisfy the inequalities (A.1). Suppose that there exists $t_1 > t_0$ such that inequality (A.2) turns into equality for $t = t_1$. Applying Lemma 1, we have that, if constant d in (A.6) is chosen sufficiently small, then estimates

$$\dot{V}\big|_{(12)} \leqslant -\frac{a_3}{2}\left(\frac{b_1(t)}{p(t)}\,\|\dot{\mathbf{q}}(t)\|^2 + r\gamma(t)p(t)(\mu+1)c_{1\sigma}\,\|\mathbf{q}(t)\|^{2\mu}\right) \leqslant -\frac{a_4}{2}\lambda(t)V^{1+\xi}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$$

are valid on the interval $[t_0, t_1]$ (except for the break points). Conditions (A.4) and (A.5) guarantee that on the interval $[t_0, t_1]$ the considered solution remains in the domains $\|(\mathbf{q}^{\mathrm{T}}, \dot{\mathbf{q}}^{\mathrm{T}})^{\mathrm{T}}\| < \Delta$ and $\|(\mathbf{q}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}\| < H$.

Then we find that

$$V^{-\xi}(t_1, \mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) \geqslant V^{-\xi}(t_0, \mathbf{q}(t_0), \dot{\mathbf{q}}(t_0)) + \frac{a_4\,\xi}{2}\,\psi(t_0, t_1) \quad \text{for} \quad k = 0,$$

$$V^{-\xi}(t_1, \mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) \geqslant (\varkappa_m \ldots \varkappa_{m-1+k})^{-\xi}\,V^{-\xi}(t_0, \mathbf{q}(t_0), \dot{\mathbf{q}}(t_0)) + \frac{a_4\,\xi}{2}\,\psi(t_0, t_1) \quad \text{for} \quad k = 1, 2, \ldots.$$

Here the values of $m = m(t_0)$ and $k = k(t_0, t_1)$ are determined in the same way as before in the study of the unperturbed system. Note that $\psi(t_0, t_1) = \psi(0, t_1) - (\varkappa_m \ldots \varkappa_{m-1+k})^{-\xi}\psi(0, t_0)$. Then one can obtain

$$\psi(0, t_1)\left(-(a_1\,\hat{c}\,D)^{-\xi} + (a_4\xi)/2\right) \leqslant (\varkappa_m \ldots \varkappa_{m-1+k})^{-\xi}\psi(0, t_0)\left(-(a_2\,\bar{c}\,D_0)^{-\xi} + (a_4\xi)/2\right).$$

The left side of this inequality is positive, while the right side is negative (see conditions (A.3)). The resulting contradiction shows that the inequality (A.2) must be preserved for all $t \geqslant t_0$. Using the proven property of the solutions of system (12), as well as their continuous dependence on the initial data, we have the required. The corollary is proved.

A.2. PROOF OF COROLLARY 3

As in the proof of Corollary 2, let us show that the constants $D_0 > 0$, $D > 0$ and $\hat{t} \geqslant 0$ can be chosen so that if the initial data of the solution $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ of systems (12) satisfy the conditions

$$t_0 \geqslant \hat{t}, \qquad \frac{1}{\widetilde{p}(t_0)}\,\|\dot{\mathbf{q}}(t_0)\|^2 + \|\mathbf{q}(t_0)\|^{\mu+1} < D_0\,\Psi^{-1/\xi}(0, t_0), \tag{A.7}$$

then

$$\frac{1}{\widetilde{p}(t)}\,\|\dot{\mathbf{q}}(t)\|^2 + \|\mathbf{q}(t)\|^{\mu+1} < D\,\Psi^{-1/\xi}(0, t) \quad \text{for} \quad t \geqslant t_0. \tag{A.8}$$

Set the specified constants, according to the conditions

$$D_0 \leqslant D, \quad -(\tilde{a}_2\,\bar{c}\,D_0)^{-\xi} + \xi < 0, \quad -(\tilde{a}_1\,\hat{c}\,D)^{-\xi} + \xi > 0, \tag{A.9}$$

$$D^{2/(\mu+1)}\Psi^{-2/(\mu-1)}(0,t) + D\,\widetilde{p}(t)\,\Psi^{-1/\xi}(0,t) < \Delta^2 \quad \text{for} \quad t \geqslant \hat{t} \geqslant \hat{T}, \tag{A.10}$$

$$D^{2/(\mu+1)}\Psi^{-2/(\mu-1)}(0,t) + D\,\Psi^{-1/\xi}(0,t) < H^2 \quad \text{for} \quad t \geqslant \hat{t} \geqslant \hat{T}, \tag{A.11}$$

$$\max\{\delta_1(t);\, \delta_2(t)\} < d_1 \quad \text{for} \quad t \geqslant \hat{t} \geqslant \hat{T}, \tag{A.12}$$

$$\max\left\{\widetilde{\delta}_1(t);\, \widetilde{\delta}_2(t)\right\} < d_2 \quad \text{for} \quad t \geqslant \hat{t} \geqslant \hat{T}, \tag{A.13}$$

where $\bar{c} = \max\{k_2;\, c_{21}\}$, $\hat{c} = \min\{k_1;\, c_{11}\}$, $H = \min\{\widetilde{H}_1;\, \widetilde{H}_2\}$, d_1 and d_2 are positive constants.

Let for the solution $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ of system (12) inequalities (A.7) be valid. Suppose that there exists $t_1 > t_0$ such that inequality (A.8) turns into equality for $t = t_1$. On the interval $[t_0, t_1]$ the considered solution remains in the domains $\|(\mathbf{q}^{\mathrm{T}}, \dot{\mathbf{q}}^{\mathrm{T}})^{\mathrm{T}}\| < \Delta$ and $\|(\mathbf{q}^{\mathrm{T}}, \widetilde{\mathbf{z}}^{\mathrm{T}})^{\mathrm{T}}\| < H$ (see conditions (A.10) and (A.11)). If constant d_1 in (A.12) is chosen sufficiently small, then, applying Lemma 1 and Jensen inequality, we find on the interval $[t_0, t_1]$ (except for the break points) the estimate

$$\dot{\widetilde{V}}\big|_{(12)} \leqslant \Big(-\Lambda(t) + A\,b_1(t)\,(\delta_1(t) + \delta_2(t))\Big)\widetilde{V}^{1+\xi}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)).$$

Here A is a positive constant (depending on the choice of D). Integrating this differential inequality on the interval $[t_0, t_1]$, one can obtain

$$\widetilde{V}^{-\xi}(t_1, \mathbf{q}(t_1), \dot{\mathbf{q}}(t_1)) \geqslant \widetilde{V}^{-\xi}(t_0, \mathbf{q}_0, \dot{\mathbf{q}}_0) + \xi\,\Psi(t_0, t_1) - A\,\xi \int\limits_{t_0}^{t_1} b_1(\tau)\,(\delta_1(\tau) + \delta_2(\tau))\,d\tau.$$

Then we have

$$\Psi(0, t_1)\left(-(\tilde{a}_1\,\hat{c}\,D)^{-\xi} + \xi - 2\,A\,\xi\,d_2\right) \leqslant \Psi(0, t_0)\left(-(\tilde{a}_2\,\bar{c}\,D_0)^{-\xi} + \xi\right). \tag{A.14}$$

If constant d_2 in (A.13) is chosen sufficiently small, then (see conditions (A.9)) the left side of inequality (A.14) is positive, while the right side is negative. The resulting contradiction shows that the inequality (A.8) must be preserved for all $t \geqslant t_0$. Using the proven property of the solutions of system (12), as well as their continuous dependence on the initial data, we have the required. The corollary is proved.

REFERENCES

1. Zubov, V.I., *Methods of A.M. Lyapunov and Their Application*, Groningen: Noordhoff, 1964.
2. Liberzon, D., *Switching in Systems and Control*, Boston, MA: Birkhauser, 2003.

3. Zhai, G., Hu, B., Yasuda, K., and Michel, A.N., Disturbance Attention Properties of Time-controlled Switched Systems, *J. of the Franklin Institute*, 2001, vol. 338, no. 7, pp. 765–779.

4. Tran, K.Q. and Yin, G., Exponential Stability of Stochastic Functional Differential Equations with Impulsive Perturbations and Markovian Switching, *Syst. Control Lett.*, 2023, vol. 173, art. no. 105457.

5. Lu, J., She, Z., Feng, W., and Ge, S.S., Stabilizability of Time-varying Switched Systems Based on Piecewise Continuous Scalar Functions, *IEEE Trans. Automat. Control*, 2019, vol. 64, no. 6, pp. 2637–2644.

6. Wang, R., Xing, J., Xiang, Z., and Yang, Q., Finite-time Stability and Asynchronously Switching Control for a Class of Time-varying Switched Nonlinear Systems, *Trans. Inst. Meas. Control*, 2020, vol. 42, no. 6, pp. 1215–1224.

7. Gao, X., Liberzon, D., Liu, J., and Basar, T., Unified Stability Criteria for Slowly Time-varying and Switched Linear Systems, *Automatica*, 2018, vol. 96, pp. 110–120.

8. Kamenetskiy, V.A., Frequency-domain Stability Conditions for Hybrid Systems, *Autom. Remote Control*, 2017, vol. 78, no. 12, pp. 2101–2119.

9. Liu, X. and Liu, D., Links Between Different Stabilities of Switched Homogeneous Systems with Delays and Uncertainties, *Int. J. Robust Nonl. Control*, 2016, vol. 26, no. 1, pp. 174–184.

10. Yang, H., Zhao, D., Jiang, B., and Ding, S., On Robust Stability of Switched Homogeneous Systems, *IET Control Theory & Applications*, 2021, vol. 15, no. 5, pp. 758–770.

11. Pesterev, A.V., Global Stability of a Second-Order Affine Switching System, *Autom. Remote Control*, 2023, vol. 84, no. 9, pp. 966–973.

12. Kozlov, V.V., On the Stability of Equilibrium Positions in Non-stationary Force Fields, *J. Appl. Math. Mech.*, 1991, vol. 55, no. 1, pp. 14–19.

13. Hatvani, L., The Effect of Damping on the Stability Properties of Equilibria of Non-autonomous Systems, *J. Appl. Math. Mech.*, 2001, vol. 65, no. 4, pp. 707–713.

14. Andreyev, A.S., The Stability of the Equilibrium Position of a Non-autonomous Mechanical System, *J. Appl. Math. Mech.*, 1996, vol. 60, no. 3, pp. 381–389.

15. Aleksandrov, A.Yu., Lakrisenko, P.A., and Platonov, A.V., Stability Analysis of Nonlinear Mechanical Systems with Switched Force Fields, *Proc. of 21th IEEE Mediterranean Conference on Control and Automation (MED'13)*, 2013, Platanias-Chania, Crite – Greece, pp. 628–633.

16. Platonov, A.V., Stability Analysis for the Lienard Equation with Discontinuous Coefficients, *Vestnik Udmurtskogo Universiteta: Matematika, Mekhanika, Komp'yuternye Nauki*, 2021, vol. 31, no. 2, pp. 226–240.

17. Platonov, A.V, On the Asymptotic Stability of Nonlinear Time-Varying Switched Systems, *Journal of Computer and Systems Sciences International*, 2018, vol. 57, no. 6, pp. 1064–2307.

18. Aleksandrov, A.Yu., Stability of Solutions of Nonlinear Systems with Unbounded Perturbations, *Mathematical Notes*, 1996, vol. 63, pp. 3–8.

This paper was recommended for publication by N.V. Kuznetsov, a member of the Editorial Board